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Dispersion of Sound in a Turbulent Flow

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Dispersion of sound in a turbulent flow is caused by the fluctuations of the velocity of the flow and by the fluctuations of the temperature, distributed irregularly in the flow. We shall assume that a turbulent flow can be described by the equations of an incompressible viscous fluid. In an incompressible fluid the dispersion by temperature irregularities and the dispersion by fluctuations of the velocity of the flow are independent. The total effect of dispersion is the sum of these effects.

Sound propagation in a turbulent flow is described by the wave equation of the acoustics of a moving medium:

$$\Delta\Psi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{v}\vec{\nabla}\right)^2 \Psi = 0, \tag{1}$$

where Ψ is the potential of the sound field, c the speed of sound, and $\mathbf{v}$ the velocity of the motion of the medium. In a system of coordinates that moves with the average velocity of the flow, $\mathbf{v}$ refers to the turbulent velocity. If we assume that the fluctuations of the velocity of the flow are considerably smaller than the speed of sound in the medium, we can disregard the term containing v^2 in Eq. (1). This equation then becomes

$$\Delta\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{2}{c^2} v_i \frac{\partial^2\Psi}{\partial x_i \partial t} + \frac{1}{c^2}\frac{\partial v_i}{\partial t}\frac{\partial \Psi}{\partial x_i}. \tag{2}$$

In this expression and hereafter we omit the summation sign from 1 to 3 for the indices that are repeated twice. In order to compare the values of the first and second terms in the right-hand side of (2), we must examine the composition of the spectrum of the velocity field of the flow. If the highest frequency contained in the spectrum of v is Ω, then $\partial v_i/\partial t$ is of the order Ωv_i; and $\partial\Psi/\partial t$ is of the order of $\omega\Psi$, where ω is the sound frequency. Thus, if the condition $\omega > \Omega$ is satisfied, we can disregard the second term in the right-hand side of Eq. (2). According to experimental data, the value of Ω does not exceed 100 cycles/sec, and the second term in the right-hand side of Eq. (2) will have an effect only for the lowest sound frequencies. (It must be noted that for the frequencies comparable with the frequencies of turbulent fluctuations the use of Eq. (1) is justified only when the fluctuations of pressure in the flow considerably exceed the sound pressures.)

The equation

$$\Delta\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{2}{c^2} v \frac{\partial^2\Psi}{\partial x_i \partial t}$$

(without the term containing the acceleration) can be solved by the method

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of successive approximations. As a first approximation, one chooses the plane monochromatic wave

$$A_0 e^{i(k_j x_j - \omega t)}$$

with the amplitude A_0, the frequency ω, and the wave vector k_j. Obukhov[1] (assuming the statistical homogeneity and isotropism of the flow) obtained the following equation for the mean square of the amplitude of the dispersion in the direction of the single wave vector **n**:

$$\overline{|\Psi_1|^2} = \frac{A_0^2 \omega^4 k_i k_j V}{(2\pi r c^2)^2} \int_V R_{ij}(\vec{\rho})\, e^{i(\mathbf{k}-k\mathbf{n})\rho}\, (d\vec{\rho}), \tag{3}$$

where V is the magnitude of the dispersed volume, r the distance from this volume to the reference point, and

$$R_{ij}(\vec{\rho}) = \overline{v_i(\mathbf{r})\, v_j(\mathbf{r}+\vec{\rho})}. \tag{4}$$

We define the z axis by the vector $\mathbf{k} - k\mathbf{n}$ and the x, z plane by the vectors $\mathbf{k}$ and $\mathbf{k} - k\mathbf{n}$. If we denote by θ the angle between $\mathbf{k}$ and $\mathbf{n}$ (the angle between the directions of the incident and dispersed waves), we obtain

$$\mathbf{k} - k\mathbf{n} = \left\{0,\ 0,\ 2k\sin\frac{\theta}{2}\right\}; \qquad \mathbf{k} = \left\{k\cos\frac{\theta}{2},\ 0,\ k\sin\frac{\theta}{2}\right\};$$
$$\vec{\rho} = \rho\{\cos\varphi\sin\psi,\ \sin\varphi\sin\psi,\ \cos\psi\} = \rho\mathbf{m}. \tag{5}$$

Here ρ, φ, and ψ are the spherical coordinates of the point $\vec{\rho}$. In terms of these coordinates, Eq. (3) becomes

$$\overline{|\Psi_1|^2} = \frac{A_0^2 \omega^4 k_i k_j V}{(2\pi r c^2)^2} \int_V R_{ij}(\vec{\rho})\, e^{2ik\rho \sin\frac{\theta}{2}\cos\psi} \rho^2 \sin\psi\, d\rho\, d\varphi\, d\psi. \tag{6}$$

But as we know (see, for instance, reference 2),

$$R_{ij}(\vec{\rho}) = (R_{rr} - R_{tt})\, m_i m_j - R_{tt}\delta_{ij}, \tag{7}$$

where $R_{rr}(\rho)$ and $R_{tt}(\rho)$ are functions only of the distance between the points in question. In order to evaluate (6) we need only the components R_{11}, R_{13}, R_{33}, since only k_1 and k_3 are different from zero. They have the form

$$R_{11}(\vec{\rho}) = (R_{rr} - R_{tt})\cos^2\varphi\sin^2\psi + R_{tt};\ R_{13}(\vec{\rho}) = (R_{rr} - R_{tt})\cos\varphi\sin\psi\cos\psi;$$
$$R_{33}(\vec{\rho}) = (R_{rr} - R_{tt})\cos^2\psi + R_{tt}. \tag{8}$$

In an incompressible fluid R_{rr} and R_{tt} are related by Karman's equation, namely

$$R_{tt} = R_{rr} + {}^1/_2\rho R_{rr}'. \tag{9}$$

If we use this relation, we can establish the form of the dependence of $\overline{|\Psi_1|^2}$ on θ (the indicatrix of dispersion) without defining the form of the function $R_{rr}(\rho)$. When we introduce the notation

$$I_{ij}(\theta) = \int_V R_{ij}(\vec{\rho})\, e^{2ik\rho\sin\frac{\theta}{2}\cos\psi} \rho^2 \sin\psi\, d\rho\, d\varphi\, d\psi, \tag{10}$$

we can write (6) as

$$\overline{|\Psi_1|^2} = \frac{A_0^2 \omega^4 k_i k_j V}{(2\pi r c^2)^2} I_{ij}. \tag{11}$$

If the functions $R_{rr}(\rho)$ and $R_{tt}(\rho)$ decrease rapidly enough with increasing ρ throughout the volume V, then the integration over ρ in (10) can be extended from 0 to ∞ (this requires that the condition $V \gg l^3$ be satisfied; here l is the distance at which the correlation function decreases considerably).

Since R_{1s} contains the factor $\cos\varphi$, it is obvious that $I_{1s} = 0$. As we shall now show, Eq. (9) implies that I_{ss} is also zero. Substituing in (10) the R_{ss} from (8) and integrating over φ and ψ, we find that

$$I_{ss} = 4\pi\int_0^\infty \left\{R_{rr}(\rho)\left[\frac{\sin\alpha\rho}{\alpha\rho} + \frac{2\cos\alpha\rho}{\alpha^2\rho^2} - \frac{2\sin\alpha\rho}{\alpha^3\rho^3}\right] - R_{tt}(\rho)\left[\frac{2\cos\alpha\rho}{\alpha^2\rho^2} - \frac{2\sin\alpha\rho}{\alpha^3\rho^3}\right]\right\}\rho^2 d\rho, \tag{12}$$

where $\alpha = 2k\sin\theta/2$. When we replace $R_{tt}(\rho)$ by $R_{rr}(\rho) + 1/2\rho R'_{rr}(\rho)$ and integrate by parts the terms containing $R'_{rr}(\rho)$, we find that $I_{ss} = 0$ if only $\lim\limits_{\rho\to\infty} \rho R_{rr}(\rho) = 0$.

Thus, $I_{1s} = I_{ss} = 0$, and hence (11) becomes

$$\overline{|\Psi_1|^2} = \frac{A_0^2\omega^2 k^2 V}{(2\pi r c^2)^2} I_{11}(\theta)\cos^2\frac{\theta}{2}. \tag{13}$$

We shall show now that $I_{11}(0) = I_{11}(\pi) = 0$. For this purpose, let us integrate the expression for $I_{11}(0) = I_{11}(\pi)$ over φ and ψ and again use Eq. (9). We find that $I_{11}(0) = I_{11}(\pi) = 0$ if only $\lim\limits_{\rho\to\infty}\rho^2 R_{rr}(\rho) = 0$.

Thus, it follows from the incompressibility features of the turbulent motion of a fluid that the indicatrix of sound dispersion has minima for $\theta = 0$ and $\theta = \pi$.

For further computations, we must determine the form of the function $R_{rr}(\rho)$. We assume for it the expression

$$R_{rr}(\rho) = {}^1/_3\,\overline{u^2}\,e^{-\rho/l}. \tag{14}$$

Here $\overline{u^2}$ is the mean-square value of the turbulence velocity of the flow, and l the scale of the correlation defining the average magnitude of the fluctuations. We can find $R_{tt}(\rho)$ from Eq. (9). Substituting these expressions in (10), we find $I_{11}(\theta)$. For $\overline{|\Psi_1|^2}$ we obtain (omitting the intermediate calculations)

$$\overline{|\Psi_1|^2} = \frac{A_0^2 V\mu^2}{6\pi l^2}\,\frac{\sin^2\theta}{\left(1+\frac{1}{2k^2l^2}-\cos\theta\right)^3}, \tag{15}$$

where $\mu^2 = \overline{u^2}/c^2$. This expression determines the indicatrix of dispersion.

As examples, Figs. 1 and 2 show two indicatrices of dispersion corresponding to the values $kl = 1/\sqrt{2}$ and $kl = 10$. When $kl \gg 1$, almost all dispersion is directed forward; the maximum dispersion is in the direction $\theta_0 = 1/\sqrt{2}kl$ and the tendency toward narrower range of dispersion increases very rapidly with the frequency. The effective range of dispersion in the solid angle $d\Omega$ per unit of the distance traversed by the incident wave is found from Eq. (15):

$$d\sigma = \frac{\mu^2}{6\pi l}\,\frac{\sin^2\theta}{\left(1+\frac{1}{2k^2l^2}-\cos\theta\right)^3}\,d\Omega. \tag{16}$$

Integrating (16) over all the values of the angles, we find the dispersion coefficient 2α:

$$2\alpha = \frac{\mu^2}{3l}\left[\frac{4k^2l^2(1+2k^2l^2)}{1+4k^2l^2} - \ln(1+4k^2l^2)\right]. \tag{17}$$

Hence, for $kl \gg 1$ we obtain

$$2\alpha \cong \frac{2}{3l}\mu^2(kl)^2. \tag{18}$$

Since at high frequencies the main part of the dispersed energy is directed forward, we may be interested here in the part directed backward. Let us find the corresponding dispersion coefficient, which we designate $2a$. When $kl \gg 1$, we have

$$2a = \frac{\mu^2}{6\pi l}\int\limits_{\pi/2}^{\pi}\frac{\sin^2\theta}{\left(1+\frac{1}{2k^2l^2}-\cos\theta\right)^2}\, 2\pi\sin\theta\, d\theta \cong \frac{\mu^2}{3l}(1-\ln 2) \approx 0{,}1\,\frac{\mu^2}{l}. \tag{19}$$

Thus the coefficient of the "backward dispersion" becomes constant at high frequencies. When we are interested in the phase relations in a wave,

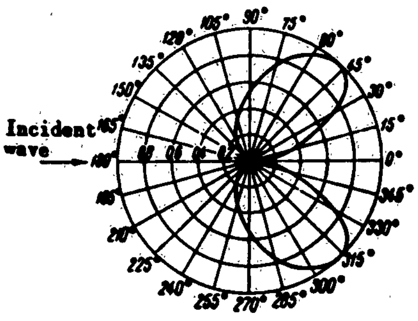

Fig. 1. $kl = 1/\sqrt{2}$.

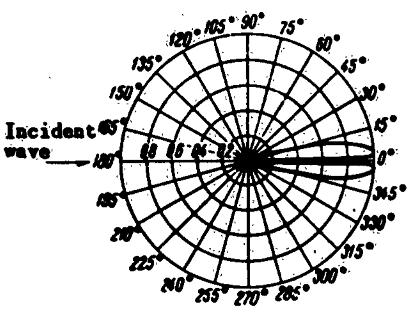

Fig. 2. $kl = 10$.

we must take the value of 2α as the dispersion coefficient (since phase relations vary in a dispersed wave, depending upon the magnitude of the wave vector, etc.). However, if the given wave interests us only from the viewpoint of energy, we must take as the dispersion coefficient a considerably smaller value, namely $2a$.

We note here that if the term allowing for the acceleration of the flow is kept in the original Eq. (2), then the following extra term appears in the expression (15) for the mean square amplitude of the dispersed wave:

$$\frac{V}{12\pi l_1 r^2}\left(\frac{\overline{w^2}}{\omega^2c^2}\right)\frac{(1-\cos\theta)^2}{\left(1+\frac{1}{2k^2l_1^2}-\cos\theta\right)^2}, \tag{20}$$

where $\overline{w^2}$ is the mean-square value of the fluctuations of acceleration and l_1 is the correlation scale of the field of accelerations. In deriving (20), the correlation function of the field of accelerations a_{ij} was taken in the form $a_{ij}(\vec{\rho}) = w_i(\vec{r})w_j(\vec{r}+\vec{\rho}) = (a_{rr} - a_{tt})\, m_i m_j + a_{tt}\delta_{ij}$ and $a_{tt}(\rho) = {}^1/_3\,\overline{w^2}\, e^{-\rho/l_1}$, while a_{rr} was determined[3] from the equation $a_{rr} = a_{tt} + \rho a_{tt}'$.

It is quite possible that the dispersion of sound by the field of accelerations may serve to explain better a damping of infrasound in the atmosphere, which damping is greater than that accounted for by the theory of dispersion through viscosity and thermal conductivity.

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